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By Maxime Bôcher.

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It is the object of the present paper to show how a certain problem in the equilibrium of particles, when treated by the use of complex quantities, can, by the introduction of homogeneous variables, be brought into connection with the theory of algebraic invariants. A very special case of this problem was considered by Gauss and F. Lucas, while another case, much more general in some respects, was made use of by Stieltjes in his discussion of polynomial solutions of certain homogeneous linear differential equations of the second order. Nowhere, however, so far as I know, has the mechanical interpretation been carried so far as I carry it here, nor has the connection with the subject of algebraic invariants been pointed out. The following pages are intended to be suggestive rather than exhaustive, only a few of the simpler applications of the method being taken up.

§ 1. The Field of Force.

Given a number of fixed particles $P_1, P_2, \ldots P_n$ in a plane, with masses $m_1, m_2, \ldots m_n$; let us suppose each of them repels with a force equal to its mass divided by the distance. We do not exclude the possibility of some of the particles having negative masses, in which case these particles will attract instead of repelling.

To obtain the field of force in the plane due to these centres of force, let us take the plane as the plane of complex numbers, and determine the positions of the particles $P_1, \ldots P_n$ in the ordinary way by means of the complex numbers $e_1, \ldots e_n$. Then, if any point P in the plane is determined by the complex number x, the force at P, due to the centres of force at $P_1, \ldots P_n$, is given both in magnitude and in direction by the complex quantity:

$$(1) \qquad K\left(\frac{m_1}{x-e_1}+\frac{m_2}{x-e_2}+\ldots+\frac{m_n}{x-e_n}\right),$$

where the symbol K is used to indicate that we are to take the conjugate of the complex quantity which follows it.*

This problem may be extended in a useful manner by the following considerations.

Let us take first the special case of two particles whose masses are the negatives of each other. Let the positions of the particles be determined by the complex quantities e_1 and e_2, and let their masses be respectively m and $-m$. Then the force at any point x in the plane is by (1):

$$m K \frac{e_1 - e_2}{(x - e_1)(x - e_2)}.$$

By considering first the angle and then the absolute value of this complex quantity we obtain at once the two results: †

I. The lines of force of two particles P_1 and P_2 whose masses are the negatives of each other are the circles through P_1 and P_2, the force at any point being directed away from the particle of positive and towards that of negative mass.

II. The intensity of the force at a point P is given by the formula:

$$\frac{m \cdot \overline{P_1 P_2}}{\overline{P_1 P} \cdot \overline{P_2 P}}.$$

The first of these statements shows that if through P_1 and P_2 we pass any spherical surface, the force is tangential to the spherical surface at all its points. The same will, therefore, be true if on any spherical surface we have any number of pairs of particles P_1, P_2; P_1', P_2'; . . ., provided that the masses of the two particles of every pair are the negatives of each other. Now it is evident that on a spherical surface any system of particles $P_1, P_2, \ldots P_n$ *the sum of whose masses is zero* is equivalent to such a system of pairs of particles. For let the masses of $P_1, P_2, \ldots P_n$ be $m_1, m_2, \ldots m_n$ respectively ($m_1 + m_2 + \ldots + m_n = 0$), and let us at an arbitrarily chosen point P_0 of the spherical surface place n coincident

* If, in particular, $m_1 = m_2 = \ldots = m_n = 1$, and if we let

$$f(x) = (x - e_1)(x - e_2) \ldots (x - e_n),$$

the field of force becomes $$K\left(\frac{f'(x)}{f(x)}\right),$$

and the points of no force are given by the roots of $f'(x) = 0$. This theorem was stated by Gauss (Werke, 3, 112) and afterwards rediscovered by F. Lucas (C. R., 1868).

† These results are very well known, and can easily be proved by other methods.

particles of masses $-m_1, -m_2, \ldots -m_n$. Since the sum of the masses of these new particles is zero, they do not affect the field of force; and now we have our pairs of particles P_1, P_0, with masses $m_1, -m_1$; P_2, P_0, with masses $m_2, -m_2$, etc. We have thus proved the theorem:

*If a number of particles lie on a spherical surface, and if the sum of their masses is zero, the force produced by them at any point on the spherical surface is tangential to that surface.**

It is this spherical field of force we now wish to study. For this purpose let P_0 be any point on the spherical surface, and project the sphere stereographically from P_0 onto the diametral plane perpendicular to the diameter through P_0. Call the projections of $P_1, \ldots P_n$ on the plane $p_1, \ldots p_n$, and let us consider the plane field of force due to particles of masses $m_1, \ldots m_n$ situated at these points.† We will prove the following two theorems:

I. *The direction of the force at any point* P *of the spherical field is the stereographic projection of the direction of the force at the corresponding point* p *of the plane field.*

Thus the lines of force on the sphere are the projections of the lines of force in the plane.

II. *The intensity of the force at any point* P *of the spherical field is equal to the intensity at the corresponding point* p *of the plane field multiplied by* $\frac{1}{2}(1 + r^2)$, *where* r *is the distance from the centre of the sphere to* p, *and the radius of the sphere is taken as the unit of length.*

We will first prove these two theorems for the special case in which there are only two particles situated at P_i and P_0 with masses m_i and $-m_i$ respectively. Since the circle $P_i P P_0$ is a line of force on the sphere, the force acting away from P_i or towards it according as m_i is positive or negative, and since the projection of this circle is the straight line $p_i p$, and this is a line of force in the plane, theorem I is obviously true in this case.

Since the intensity of the force at p is

$$f = \frac{|m_i|}{\overline{p_i p}^2},$$

* The same reasoning shows that if a number of particles lie on a circle, and if the sum of their masses is zero, the force produced by them at any point on the circle is tangential to the circle. It also admits of immediate application to spherical multiplicities in space of n dimensions.

† If one of the points P_i lies at P_0, the corresponding point p_i will be at infinity, and the particle in question in the plane is simply to be omitted.

and, by a formula obtained above, the intensity of the force at P is

$$F = \frac{|m_i| \overline{P_0 P_i}}{\overline{P_0 P} \cdot \overline{P_i P}},$$

we see that

$$F = \frac{\overline{p_i p} \cdot \overline{P_0 P_i}}{\overline{P_0 P} \cdot \overline{P_i P}} \cdot f.$$

Now it is well known that the stereographic projection of the sphere on the plane is equivalent to an inversion with regard to a sphere having P_0 as centre and radius $\sqrt{2}$. Accordingly:

$$P_0 P = \frac{2}{P_0 p} = \frac{2}{\sqrt{1 + r^2}}.$$

Moreover, by a familiar property of inversion, the triangles $P_0 P_i P$ and $P_0 p p_i$ are similar, so that:

$$\frac{p_i p}{P_i P} = \frac{P_0 p}{P_0 P_i}.$$

Substituting these values we find

$$F = \frac{1 + r^2}{2} f,$$

as was to be proved.

Having thus established theorems I and II in the special case of two particles, one of which lies at P_0, the proof in the general case follows by replacing the n particles $P_1, \ldots P_n$ as we did above, by the n *pairs* of particles P_1, P_0; P_2, P_0; $\ldots$ P_n, P_0. We thus get n forces acting at P and n corresponding forces acting at p. Moreover each force at P acts in the direction which is the stereographic projection of the direction of the corresponding force at p; and accordingly, since angles are not changed by stereographic projection, the figure formed by the *directions* of the forces at P is congruent with the figure formed by the directions of the forces at p. Moreover every force at P is $\frac{1}{2}(1 + r^2)$ times the corresponding forces at p. Accordingly the forces at P are represented by n vectors, which form a figure similar (in the geometrical sense of the word) to that formed by the n vectors which represent the forces at p. If we now complete these figures by constructing the resultant in each case, the figures will clearly remain similar. Accordingly the directions of these resultants, since they make

equal angles with corresponding components, are the stereographic projections of each other; and the intensities of the resultants are in the same ratio as the intensities of the components. Thus theorems I and II are proved.

It has long been a familiar fact that one and the same plane vector field gives the force due to n particles $p_1, p_2, \ldots p_n$ in a plane, of masses $m_1, m_2, \ldots m_n$; and the steady flow in a plane conducting lamina due to electrical sources at $p_1, \ldots p_n$ of intensities $m_1, \ldots m_n$. On the other hand it is well known that the vector field on a spherical surface which represents the steady flow of electricity on the surface due to n sources $P_1, \ldots P_n$ with intensities $m_1, \ldots m_n$ $(m_1 + \ldots + m_n = 0)$ may be obtained by projecting stereographically onto a diametral plane and considering the flow due to sources at $p_1, \ldots p_n$ (the projections of $P_1, \ldots P_n$) of intensities $m_1, \ldots m_n$. If the length of each vector in this last mentioned plane field be multiplied by $\frac{1}{2}(1 + r^2)$, — the radius of the sphere being taken as unit of length, — and the field thus modified be projected back onto the sphere, we obtain the desired flow on the sphere. Comparing these facts with theorems I and II we see that *one and the same vector field on the spherical surface represents the flow in the electrical problem just mentioned and the force in the mechanical problem considered before.*

§ 2. Relation to Algebraic Invariants.

Let us now, in the discussion of the field of force, introduce homogeneous variables by the formulae:

$$x = \frac{x_1}{x_2}, \qquad e_i = \frac{e_i'}{e_i''}.$$

The plane field (1) then becomes:

$$K\left[x_2 \sum_1^n \frac{m_i e_i''}{e_i'' x_1 - e_i' x_2}\right].$$

In this formula we may, without real loss of generality, assume that $m_1 + m_2 + \ldots + m_n = 0$. For if this were not the case we could regard our formula as the special case of the corresponding formula where n is replaced by $n + 1$, $e_{n+1}'' = 0$, and m_{n+1} is so chosen that $m_1 + \ldots + m_{n+1} = 0$. This expression for the field of force, when reduced to a common denominator, becomes, when we take account of the relation $m_1 + m_2 + \ldots + m_n = 0$ which we now assume to hold,

$$(2) \qquad K\left[x_2^2 \frac{\phi(x_1, x_2)}{(e_1'' x_1 - e_1' x_2) \ldots (e_n'' x_1 - e_n' x_2)}\right],$$

where ϕ is a binary form of degree $n - 2$. We will now prove that ϕ is a covariant of weight 1 of the system of linear forms which stand in the denominator of (2).

For this purpose let us notice that in the case $n = 2$, ϕ reduces at once to $m_1 (e_1' e_2'' - e_2' e_1'')$ which is an invariant of weight 1. If then we regard the particle e_n of mass m_n as consisting of $n - 1$ coincident particles of masses $- m_1, - m_2, \ldots - m_{n-1}$, we are led to write (2) in the form

$$K\left[x_2^2 \sum_{i=1}^{i=n-1} \frac{m_i (e_i' e_n'' - e_n' e_i'')}{(e_i'' x_1 - e_i' x_2)(e_n'' x_1 - e_n' x_2)}\right].$$

Each term under the sign of summation being a covariant of weight 1, the same is true of their aggregate, which is merely the fraction in (2) of which ϕ is the numerator. The denominator of this fraction being a covariant of weight zero, it follows that its numerator ϕ is a covariant of weight 1, as was to be proved.

The form ϕ obviously vanishes at every point in the field where the intensity of the force is zero. Besides these points of equilibrium ϕ cannot vanish anywhere except perhaps at the points e_i where the denominator of (2) vanishes. If a particle e_i whose mass is different from zero does not coincide with any of the other particles, ϕ will not vanish there; for if it did, formula (2) would yield a force whose intensity remains finite as we approach e_i. If, however, k of the particles whose total mass is not zero coincide at the point e_i, the form ϕ must contain the linear factor $e_i'' x_1 - e_i' x_2$ exactly $k - 1$ times in order that (2) should yield a force which becomes infinite to the first order at e_i. Such a point, in spite of the vanishing of ϕ, is, of course, not a point of equilibrium. We shall, however, speak of it as a point of *pseudo-equilibrium*, in justification of which term we may notice that if we let k distinct particles whose total mass is not zero coincide at e_i, $k - 1$ points of true equilibrium fall together at this point.

It remains then only to consider a point at which k of the particles whose total mass is zero coincide. At such a point we have a force of finite intensity and, accordingly, ϕ must have at least a k-fold root there. If the multiplicity is exactly k, we do not have equilibrium, but such a point we shall again call a point of pseudo-equilibrium. If the multiplicity of the root of ϕ is greater than k we have true equilibrium.

We have so far regarded ϕ as an integral rational covariant of weight 1 of the system of linear forms

$$(3) \qquad e_i'' x_1 - e_i' x_2 \qquad (i = 1, 2, \ldots n)$$

which involves n real parameters $m_1, \ldots m_n$ connected by the relation $m_1 + m_2 + \ldots + m_n = 0$.*

Instead, however, of determining the positions of the particles by the vanishing of the linear forms (3), as we have practically been doing, we may, if we prefer, build up forms of higher degree $f_1, f_2, \ldots f_k$ by multiplying the linear forms (3) together in groups, and then regard the positions of the particles as determined by the vanishing of these forms f. The form ϕ will still be a covariant of the system of forms f, but it will in general be an *irrational* covariant of this system. If, however, the particles corrresponding to the form f_i all have the same mass, ϕ is rational and symmetric in the roots of f_i and therefore rational in the coefficients of f_i. Thus we obtain the result:

Given a system of binary forms $\mathrm{f}_1, \mathrm{f}_2, \ldots \mathrm{f}_k$ *of degrees* $\mathrm{p}_1, \mathrm{p}_2, \ldots \mathrm{p}_k$ *respectively, and* k *real quantities* $\mathrm{m}_1, \mathrm{m}_2, \ldots \mathrm{m}_k$ *subject to the condition* $\mathrm{p}_1 \mathrm{m}_1 + \ldots + \mathrm{p}_k \mathrm{m}_k = 0$; *and suppose that at each of the* p_i *points determined in the complex plane or on the complex sphere by the equation* $\mathrm{f}_i = 0$ *are placed equal particles each of mass* m_i *which repel with a force which varies directly as the mass and inversely as the distance; then the positions of equilibrium in this plane or spherical field of force are determined as the roots of a certain integral rational covariant* ϕ *of the system of forms* f.

Besides vanishing at the points of equilibrium, ϕ *vanishes only at the points of pseudo-equilibrium described above. Such points can occur only at the multiple roots of a form* f_i *or at a common root of two of these forms.*

The simplest case is when $k = 2$. Here $p_1 m_1 + p_2 m_2 = 0$, and therefore, since only the ratio of the masses m_1 and m_2 is important, we may without real loss of generality assume that $m_1 = p_2$, $m_2 = -p_1$. If we let:

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* It would be possible to consider the more general case in which the quantities m_i are complex, the force having then not merely an attractive or repulsive component, but also a component at right angles to this. Thus if the quantities m_i are pure imaginaries, we have, in the plane, the electromagnetic field due to the steady flow of electric currents through long straight wires which cut the plane at right angles.

$$f_1 = (e_1'' x_1 - e_1' x_2) \ldots (e_{p_1}'' x_1 - e_{p_1} x_2),$$
$$f_2 = (e_{p_1+1}'' x_1 - e_{p_1+1}' x_2) \ldots (e_{p_1+p_2}'' x_1 - e_{p_1+p_2}' x_2),$$

the field of force is

$$K\left[x_2\left(p_2 \sum_{i=1}^{i=p_1} \frac{e_i''}{e_i'' x_1 - e_i' x_2} - p_1 \sum_{i=p_1+1}^{i=p_1+p_2} \frac{e_i''}{e_i'' x_1 - e_i' x_2}\right)\right] =$$
$$K\left[x_2\left(p_2 \frac{\frac{\partial f_1}{\partial x_1}}{f_1} - p_1 \frac{\frac{\partial f_2}{\partial x_1}}{f_2}\right)\right].$$

Accordingly

$$\phi = \frac{1}{x_2}\left(p_2 f_2 \frac{\partial f_1}{\partial x_1} - p_1 f_1 \frac{\partial f_2}{\partial x_1}\right),$$

which reduces, when we apply Euler's theorem for homogeneous functions, to the Jacobian of f_1 and f_2. Hence

The vanishing of the Jacobian of two binary forms f_1 *and* f_2 *of degrees* p_1 *and* p_2 *respectively determines the points of equilibrium in the field of force due to* p_1 *particles of mass* p_2 *situated at the roots of* f_1, *and* p_2 *particles of mass* $-p_1$ *situated at the roots of* f_2.*

It is easy now to express the covariant ϕ which we have in the general case integrally and rationally in terms of the ground-forms and their Jacobians. For this purpose let us write as the field of force:

$$K\left[x_2 \sum_{i=1}^{i=k} m_i \frac{\frac{\partial f_i}{\partial x_1}}{f_i}\right] = K\left[\frac{x_2}{p_k} \sum_{i=1}^{i=k-1} m_i\left(p_k \frac{\frac{\partial f_i}{\partial x_1}}{f_i} - p_i \frac{\frac{\partial f_k}{\partial x_1}}{f_k}\right)\right]$$
$$= K\left[\frac{x_2^2}{p_k} \sum_{i=1}^{i=k-1} \frac{m_i J_{ik}}{f_i f_k}\right],$$

where J_{ik} denotes the Jacobian of f_i and f_k. Accordingly:

$$\phi = \sum_{i=1}^{i=k-1}\left(\frac{m_i}{p_k} J_{ik} \cdot f_1 \ldots f_{i-1} f_{i+1} \ldots f_{k-1}\right),$$

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* If one of the two ground-forms is linear, the theorem may be stated thus:

If ϕ is the first polar of the point (y_1, y_2) with regard to a binary form f of degree p, the vanishing of ϕ determines the points of equilibrium in the field of force due to p unit particles situated at the roots of f, and one particle of mass $-p$ at, (y_1, y_2).

The special case of this where $y_2 = 0$ leads us back to Gauss's theorem referred to near the beginning of § 1.

an expression from which the lack of symmetry could be removed by well known and obvious methods.*

The mechanical significance we have thus attached to the vanishing of certain covariants makes it sometimes possible by direct mechanical intuition to obtain information concerning the position of the roots of these covariants. The following theorem serves as a starting point:

A point (in the plane or on the sphere) cannot be a position of true equilibrium if it is possible to draw a circle through it upon which not all the particles lie, and which completely separates the attractive particles which do not lie on it from the repulsive particles which do not lie on it.

This is at once obvious when we consider the spherical field of force, for there will clearly be in this case a component at the point in question perpendicular to the circle.

This principle enables us in many cases, when we know the positions of the roots of the ground-forms, to find regions in the plane in which no root of the covariant ϕ lies.

Let us look at the case of two ground-forms f_1 and f_2; and let us suppose that there are two regions S_1 and S_2 on the sphere, bounded by circles C_1 and C_2 respectively, which do not overlap, and such that all the roots of f_1 lie within or on the boundary of S_1 and all the roots of f_2 lie within or on the boundary of S_2; provided, however, that if C_1 and C_2 have a point or points in common, not all the roots of both f_1 and f_2 shall lie at such common points. In this case the principle just stated shows that the Jacobian of f_1 and f_2 can have no root lying in the region between the circles C_1 and C_2. In other words all the roots of this Jacobian must lie within or on the boundary of S_1 and S_2.

We can in general go further by constructing in S_1 and S_2 curvilinear polygons T_1 and T_2 bounded by arcs of circles such that each side of T_i passes through at least two roots of f_i and, when extended, through at least one root of the other ground-form, and is so situated that the circle of which it forms a part separates from one another all the roots of f_1

* We thus get as the field of force:

$$K\left[\frac{x_2^2}{k}\sum_{i=1}^{i=k}\sum_{j=1}^{j=k}\frac{m_i J_{ij}}{p_i f_i f_j}\right],$$

and the covariant becomes:

$$\phi = \frac{1}{k}\sum_{i=1}^{i=k}\sum_{j=1}^{j=k}\frac{m_i}{p_i}\frac{f_1 \cdot \cdot \cdot f_k}{f_i f_j}J_{ij},$$

where we must remember that $J_{ii} = 0$, $J_{ij} = -J_{ji}$.

and f_2 which do not lie upon it. Some of the roots of f_i will form the vertices of T_i, while the others lie within it or on its boundary. It follows by the principle we have already made use of, that the roots of the Jacobian must lie in the polygons T_1 and T_2.

By a slight additional consideration we can even determine how the roots of the Jacobian are distributed between the two regions. For this purpose let us allow the roots of f_1 and f_2 to change, always remaining in S_1 and S_2 respectively, in such a way that the roots of f_i all approach a point a_i within S_i. If this change is a continuous one, the roots of the Jacobian will also change continuously on the complex sphere, and therefore such of these roots as originally lay in S_i will remain there. We may, however, during this process allow the regions S_1 and S_2 to shrink down towards the points a_1 and a_2 respectively, while they always include the roots of f_1 and f_2. Accordingly all the roots of the Jacobian which originally lay in S_i must be approaching a_i as their limit. But we have seen above that when k particles whose total mass is not zero fall together at a point, $k - 1$ positions of true equilibrium coalesce into a position of pseudo-equilibrium. The Jacobian must therefore have had just $p_1 - 1$ roots in S_1 and $p_2 - 1$ roots in S_2. Hence the theorem:

If the roots of a binary form f_1 *of degree* p_1 *lie within or on the boundary of a region* T_1, *and if the roots of a second binary form* f_2 *of degree* p_2 *lie within or on the boundary of a second region* T_2 *which does not overlap or touch the first, and if these two regions are bounded by arcs of circles each one of which circles separates the roots of* f_1 *which do not lie on it from the roots of* f_2 *which do not lie on it; then the Jacobian of* f_1 *and* f_2 *has just* $\mathrm{p}_1 - 1$ *roots in* T_1 *and* $\mathrm{p}_2 - 1$ *roots in* T_2.*

The method of proof here used can be immediately extended to the general case of k ground-forms $f_1, \ldots f_k$. If all the roots of such of these forms as correspond to the positive constants of the set $m_1, m_2, \ldots m_k$ lie in T_1 and all the roots of the other forms lie in T_2, we see in this way that all the roots of the covariant ϕ lie in T_1 or T_2, and that ϕ has in T_i one less root than the ground-forms have there.

The principle we have used so far is not the only one which helps us

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* The special case in which one of the ground-forms reduces to x_2 gives the following theorem, which is an immediate consequence of Gauss's theorem quoted above, and was first explicitly stated by F. Lucas, Journal de l'École Polytechnique, Cahier 46 (1879), p. 8.

The roots of the derivative of any polynomial in x lie in any convex rectilinear polygon in the complex plane which includes within itself or on its perimeter all the roots of the original polynomial.

to locate the roots of ϕ. Let us, for simplicity, consider again merely the case of two ground-forms f_1 and f_2, and let us suppose that these forms are real, so that the roots of each, so far as they are not real, will be conjugate imaginary in pairs. The force at any point on the axis of reals will here be in the direction of this axis, and we have the theorem:

If ϕ is the Jacobian of two real binary forms f_1 *and* f_2, *then in an interval of the axis of reals bounded by roots of one of these ground-forms and containing no root of either form there lies at least one root of* ϕ.

The interval in question may, it should be noticed, be infinite, extending through the point at infinity if this is not a root of f_1 or f_2, otherwise extending up to this point. Immediate consequences of the last theorem are these:

If all the roots of f_1 *and* f_2 *are real, their Jacobian has a number of real roots at least as great as the difference between the degrees of* f_1 *and* f_2.

If all the roots of f_1 *and* f_2 *are real and distinct, and if all the roots of one of these forms lie in one of the intervals into which the roots of the other form divide the axis of reals, then all the roots of the Jacobian of* f_1 *and* f_2 *are real and distinct, and just one of these roots lies in each of the intervals into which the roots of* f_1 *and* f_2 *divide the axis of reals, except that the two intervals which are bounded at one end by a root of* f_1, *at the other by a root of* f_2, *contain no root of the Jacobian.*

These theorems also can easily be extended to the case of more than two ground-forms.

The proofs of the theorems which we have here deduced from mechanical intuition can readily be thrown, without essentially modifying their character, into purely algebraic form. The mechanical problem must nevertheless be regarded as valuable, for it suggests not only the theorems but also the method of proof.

§ 3. Stieltjes's Generalization.

Up to this point we have been considering the problem of determining the positions in which a single particle free to move in a certain field of force can rest in equilibrium. This problem may be generalized in a fruitful manner by considering with Stieltjes * a number of particles of

* Acta Math., 6 (1885), 323, where, however, only the case in which all fixed and movable particles lie on the axis of reals is considered. The case in which all the particles are free to lie anywhere in the complex plane was taken up for the first time in the book of the present writer entitled, Ueber die Reihenentwickelungen der Potentialtheorie, Leipzig, 1894, p. 215. Cf. also Bull. Amer. Math. Soc., March, 1898, p. 256.

equal mass which are all free to move in the plane or on the sphere, and are acted upon not merely by the field of force due to the fixed particles, but also by one another, the action here again being repulsion of intensity equal to the product of the masses divided by the distances. We may without loss of generality take the mass of each of the moving particles as unity, in which case the total mass of the fixed and moving particles must evidently be +1 in order that the force acting on each of the moving particles should be tangential to the sphere.* The problem of determining the positions of equilibrium of the system of moving particles is readily seen, by reference to the results already obtained, to have the same invariant character as in the special case already considered. Neither Stieltjes's original treatment of the problem nor my own earlier methods brought out this fact in the analytic work, the positions of equilibrium being obtained as the roots of the polynomial solutions of certain homogeneous linear differential equations of the second order, — equations whose invariant character was in no way evident. I hope to take up this whole subject before long from the point of view here indicated. I content myself here with pointing out, by a simple example, how Stieltjes's problem can be brought into connection with the elementary theory of algebraic invariants.

If f *is a binary cubic with distinct roots, and if three fixed particles, each of mass* $\frac{1}{3}$ (1 — k) *are situated at the roots of* f, *then the position of equilibrium, if any exists, of* k *movable particles of mass* +1 *is given by the vanishing of a certain covariant* ϕ *of* f, *which in the simplest cases* k $\overline{\underline{<}}$ 7 *is:*

$$\begin{array}{ll} k=2 & \phi = H, \\ k=3 & \phi = J, \\ k=4 & \text{there is no position of equilibrium,} \\ k=5 & \phi = H\cdot J, \\ k=6 & \phi = \Delta f^2 - 7H^3, \\ k=7 & \text{there is no position of equilibrium.} \end{array}$$

Here Δ and H denote the discriminant and Hessian of f, and J the Jacobian of f and H, where, however, in order that the formula for $k=6$ be correct, we must suppose that, f being written:

$$f = a_0 x_1^3 + 3a_1 x_1^2 x_2 + 3a_2 x_1 x_2^2 + a_3 x_2^3,$$

* This restriction is not necessary when, as has always been the case heretofore, the *plane* problem only is considered, and no attempt is made to bring out its invariant character.

numerical factors are included in Δ and H so as to make them have integral coefficients without common factors.

This problem of Stieltjes admits of generalization in still another direction by considering not the field of force due to a number of fixed particles which repel according to the law of the inverse distance, but the field of force given in the plane by the formula

$$K[f(x)],$$

where $f(x)$ is any analytic function of the complex variable x.* We should here seek the positions of equilibrium of a group of k unit particles which repel one another with a force equal to the reciprocal of the distance. Denoting the complex quantities which determine the positions of these movable particles by $x_1, x_2, \ldots x_k$, and letting

$$\phi(x) = (x - x_1)(x - x_2) \ldots (x - x_k),$$

the equations of equilibrium are

$$\tfrac{1}{2}\frac{\phi''(x_i)}{\phi'(x_i)} + f(x_i) = 0. \qquad (i = 1, 2, \ldots k).$$

Let us consider here in more detail the special case in which

$$f(x) = - c\,x,$$

where c is a positive constant. The force here may be described as attraction towards the axis of imaginaries and repulsion from the axis of reals, the intensity of the force being in both cases directly proportional to the distance from the axis in question, and the factor of proportionality being the same in the two cases. It is clear that the particles cannot be in equilibrium unless they all lie on the axis of reals, and it is equally clear that there is at least one position of equilibrium of this latter sort. In order to find it write the equations of equilibrium in the form

$$\phi''(x_i) - 2\,c\,x_i\,\phi'(x_i) = 0, \qquad (i = 1, 2, \ldots k).$$

These equations show that the polynomials $\phi(x)$ and $\phi''(x) - 2\,cx\phi'(x)$,

* My colleague, Professor B. O. Peirce, calls my attention to the fact that this amounts to considering the field of force which has the real part of $\int f(x)\,dx$ as a force function. We are thus dealing with the most general field, which has a force function with continuous first and second partial derivatives and satisfies Laplace's equation.

which are of the same degree, have the same roots (obviously distinct, since no two particles could coincide in a position of equilibrium), so that they differ only by a constant factor. Accordingly the polynomial ϕ, whose roots determine a position of equilibrium, satisfies a differential equation of the form:

$$\frac{d^2 y}{dx^2} - 2cx\frac{dy}{dx} = Cy.$$

Substituting for y the polynomial ϕ, we find at once by comparing the coefficients of x^k on both sides that $C = -2ck$. Hence:

The positions of equilibrium of a group of k *unit particles repelling one another inversely as the distance, and situated in the field of force* K $(-cx)$, *are the roots of the polynomials of the* k*th degree which satisfy the differential equation*

$$\frac{d^2y}{dx^2} - 2cx\frac{dy}{dx} + 2cky = 0. \tag{4}$$

The mechanical problem shows that this equation must have at least one polynomial solution of the kth degree, a fact which may be readily verified by substituting in (4) a polynomial with undetermined coefficients. This substitution shows that there is only one polynomial solution.

The polynomials thus obtained are known as Hermite's Polynomials, having been first discussed by this mathematician * from the following point of view. Consider the function e^{-cx^2}. If we differentiate this function k times with regard to x we obtain the original function multiplied by a certain polynomial in x. Among the properties of this function which Hermite develops is the fact that it satisfies the differential equation (4). This shows its identity with our polynomial ϕ. The fact that all the roots of ϕ are real and distinct, which followed from our mechanical problem, follows also at once from Hermite's definition

If we make the transformation

$$z = e^{-\frac{cx^2}{2}} \cdot y,$$

the differential equation (4) takes the form

$$\frac{d^2z}{dx^2} = [c^2x^2 - c(2k+1)]z, \tag{5}$$

* Comptes Rendus, 1864, pp. 93 and 266.

and this is the equation for the functions of the parabolic cylinder in the form most commonly used. That in the special case we have just been considering (the case in which k is a positive integer) this equation has a solution in the form of the product of an exponential function and a polynomial was noticed by K. Baer.* I am not aware, however, that the connection between this fact and Hermite's work on the one hand and Stieltjes's method on the other has ever been pointed out.

The field of force $K(-cx)$ which we have just considered may be regarded as the limit of the field due to two particles of mass m situated at the points e and $-e$ and two particles of mass $-m$ situated at ie and $-ie$, each particle repelling directly as the mass and inversely as the distance, as e and m both become infinite. That is, the field may be regarded as due to a *quadruplet* at infinity. A simpler case, from this point of view, would be that of a *doublet* at infinity which would produce a uniform field, let us say $K\left(-\frac{c}{2}\right)$. Here, however, there is clearly no position of equilibrium. In order to make equilibrium possible, let us introduce into this uniform field a single fixed particle of mass m, repelling inversely as the distance, and situated, say, at the origin. This problem can be carried through precisely as was the one above, and leads us, not to equation (5), but to:

$$(6)\qquad 4x\frac{d^2z}{dx^2} + 8m\frac{dz}{dx} = [c^2x - 4c(m+k)]z,$$

and we see again that if k is a positive integer, (6) has a solution which is the product of an exponential factor and a polynomial, the roots of this polynomial being the positions of equilibrium in the problem last considered.

It is interesting to note that equation (6), in the special case $m = \frac{1}{4}$, reduces to the form in which the equation for the functions of the parabolic cylinder presents itself when the subject is approached from a broader point of view than is ordinarily done.†

We may, of course, consider doublets, triplets, etc., which do not lie at infinity, and such centres of force may come in in any number along-

* Programm, Küstrin, 1883, p. 9.

† Cf. the book of the present writer already referred to. By introducing x^2 in place of x as the independent variable in (5), and $2k$ in place of k, we can pass at once to the special case of equation (6) in which $m = \frac{1}{4}$.

side of the simple repulsive particles. Moreover the invariant character of these problems can be brought out both by using the sphere instead of the plane, and by using homogeneous variables and throwing the analytical work into invariant form. I hope, as I have already said, to come back in detail to all these questions.

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